

UNITARY ORBITS OF SELF-ADJOINT OPERATORS IN SIMPLE \mathcal{Z} -STABLE C^* -ALGEBRAS

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ABSTRACT. We prove that in a simple, unital, exact, \mathcal{Z} -stable C^* -algebra of stable rank one, the distance between the unitary orbits of self-adjoint elements with connected spectrum is completely determined by spectral data. This fails without the assumption of \mathcal{Z} -stability.

1. INTRODUCTION

It is well known that the focus of the Elliott classification programme has shifted from the class of all separable nuclear C^* -algebras to, in light of the counterexamples [33] and [25], those that are sufficiently well behaved. It has subsequently become a natural pursuit to unify the various notions of ‘regularity’ for C^* -algebras, most notably in the form of the Toms–Winter conjecture (see for example [35]). Furthermore, any meaningful notion of regularity entails not only amenability to classification but also, inevitably, tameness of internal structure. In particular, there is a persistent expectation of mimicry in regular C^* -algebras of the good behaviour found in the world of von Neumann factors (see especially [27] for an exposition of this point of view). This article further develops that theme.

One of the most fundamental questions one can ask in an operator algebra is: when are two operators unitarily equivalent? For normal operators, this problem is inextricably linked with spectral theory, and we refer to [28] for a discussion of its history. The present article addresses the related problem, which has an equally storied past, of computing the distance d_U between unitary orbits in terms of spectral information. For normal matrices $a, b \in M_n$, one asks whether $d_U(a, b)$ is equal to the *optimal matching distance*

$$\delta(a, b) = \min_{\sigma \in S_n} \max_{1 \leq i \leq n} |\alpha_i - \beta_{\sigma(i)}| \quad (1.1)$$

where $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n are the eigenvalues of a and b respectively. If a and b are self-adjoint then a classical result of H. Weyl [34] says that these distances do indeed agree (and moreover $\delta(a, b)$ can be measured by listing the eigenvalues in ascending order). This continues to hold for example for unitary matrices [3] but not necessarily for normal matrices [16] (although d_U and δ are known in general to be Lipschitz equivalent [4]).

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Redefining δ as a *crude multiplicity function* described in terms of ranks of spectral projections, this was extended in [2] to self-adjoint operators on infinite dimensional Hilbert space, and further in [15] to self-adjoint elements in a σ -finite semifinite factor. In the latter case, one defines $\delta(a, b)$ for normal elements a and b to be the infimum over $r > 0$ such that

$$\tau(\chi_{U_r}(a)) \geq \tau(\chi_U(b)) \quad \text{and} \quad \tau(\chi_{U_r}(b)) \geq \tau(\chi_U(a)) \quad (1.2)$$

for every open subset $U \subset \mathbb{C}$. Here τ is a fixed faithful normal semifinite trace, $U_r = \{t \mid \text{dist}(t, U) < r\}$ and χ_U is the indicator function on the open subset $U \subset \mathbb{C}$.

The C^* -regularity property most relevant to this article is that of ‘ \mathcal{Z} -stability’, that is, tensorial absorption of the Jiang–Su algebra \mathcal{Z} [18]. A simple, unital, \mathcal{Z} -stable C^* -algebra is either stably finite or purely infinite [14]. In the latter case, it is shown in [29] that (modulo K -theory) the distance $d_U(a, b)$ between the normal operators a and b is simply the Hausdorff distance between their spectra. This mirrors the corresponding result obtained for type III factors in [15], and this article may therefore be regarded as a step towards completion of the analogy. The appropriate C^* -analogue of (1.2) is the Lévy–Prokhorov distance d_P discussed in Section 5 below, and we do indeed obtain that $d_U(a, b) = d_P(a, b)$ for self-adjoint elements with connected spectra in the stably finite setting. (By translating by a multiple of the unit, there is no loss of generality in restricting to positive elements.)

Our strategy along the way is to exploit another notion of spectral distance, the pseudometric d_W defined in [10] in terms of Cuntz equivalence (see Section 2), and to bootstrap the equality $d_U = d_W$ from matrices to a class of algebras that exhausts the tracial invariant (Section 3) and ultimately via classification to the class of simple \mathcal{Z} -stable C^* -algebras of stable rank one (Section 4), a class for which we also obtain $d_W = d_P = d_U$ (Section 5).

Finally, it should be noted that the assumption of connected spectra is necessary for a purely measure theoretic calculation of d_U . There exist, for example, nonequivalent projections in a simple, unital, monotracial AF algebra that have the same trace (see [5, 7.6.2]). Moreover, this computation really does rely on regularity of the ambient C^* -algebra. See [24, Section 4.3] for an example of a simple (necessarily non- \mathcal{Z} -stable) AH algebra where d_U and d_P differ: there exist full spectrum positive contractions a, b that are not approximately unitarily equivalent, so $d_U(a, b) \neq 0$, but for which $d_W(a, b) = 0$ (hence also $d_P(a, b) = 0$).

2. PRELIMINARIES

Let A be a C^* -algebra. We will denote the cone of positive elements in A by A^+ , the minimal unitisation of A by \tilde{A} , and the group of unitaries in \tilde{A} by $\mathcal{U}(\tilde{A})$.

Given a positive element $x \in A$, let $e_s(x) = (x - s)_+$, that is, the element of $C^*(x)$ corresponding under functional calculus to $f(t) = \max\{0, t - s\}$. Two positive elements $x, y \in A$ are Cuntz subequivalent, written $x \precsim y$ if $\|x - v_n y v_n^*\| \rightarrow 0$ for some $v_n \in A \otimes \mathcal{K}$, and denote by \sim the relation that symmetrises \precsim . (We refer the reader to [1] for more information on the Cuntz semigroup $\text{Cu}(A) = (A \otimes \mathcal{K})^+ / \sim$.)

We consider the following pseudometrics on the set of positive elements of A :

$$d_U(a, b) = \inf\{\|uau^* - b\| \mid u \in \mathcal{U}(\tilde{A})\} \quad (2.1)$$

(so $d_U(a, b) = 0$ if and only if a and b are approximately unitarily equivalent), and

$$d_W(a, b) = \inf\{r > 0 \mid e_{t+r}(a) \precsim e_t(b) \text{ and } e_{t+r}(b) \precsim e_t(a) \forall t > 0\}. \quad (2.2)$$

It is perhaps an instructive exercise to convince oneself that (1.1) and (2.2) agree for positive matrices.

The following, which implies in particular that d_U and d_W are continuous, makes its initial appearance as [10, Corollary 9.1]. See [24, Lemma 1] for a straightforward functional calculus proof.

LEMMA 2.1. *For $a, b \in A^+$, we have $d_W(a, b) \leq d_U(a, b) \leq \|a - b\|$.*

The main result of [10] is that, if A has stable rank one, then

$$d_W(a, b) \leq d_U(a, b) \leq 8d_W(a, b). \quad (2.3)$$

In particular, two $*$ -homomorphisms from $C_0(0, 1]$ to A are approximately unitarily equivalent if and only if they agree at the level of the Cuntz semigroup. This inequality (with the factor of 8 improved to 4) has been demonstrated in [24] for a class of C^* -algebras strictly larger than that of the stable rank one algebras, provided that d_U is computed in the stabilisation (note that this does not make a difference in the stable rank one case).

We are interested in showing that $d_U(a, b) = d_W(a, b)$ for every $a, b \in A^+$ when A is sufficiently well behaved. This is the case, for example for inductive limits $\varinjlim C(X_i)$, where the X_i are compact Hausdorff spaces of topological dimension at most 2 with $\check{H}^2(X_i) = 0$ (see [24, Proposition 5]).

REMARK 2.2. By Lemma 2.1, to show that $d_U = d_W$ it suffices to find a dense subset A' of A^+ such that $d_U(a, b) \leq d_W(a, b)$ for every $a, b \in A'$.

LEMMA 2.3. *The property “ $d_U = d_W$ ” passes to finite direct sums and sequential inductive limits.*

Proof. See [24, Lemma 5] for inductive limits. Finite direct sums are straightforward. \square

We will first show, in Section 3, that certain stably projectionless algebras satisfy $d_U = d_W$. We then use classification in Section 4 to obtain this for simple, \mathcal{Z} -stable algebras for positive contractions with full spectrum.

3. RAZAK BLOCKS

A *Razak block* is a C^* -algebra of the form

$$\{f \in C([0, 1], M_k \otimes M_n) \mid f(0) = c \otimes 1_{n-1}, f(1) = c \otimes 1_n, c \in M_k\}, \quad (3.1)$$

and is regarded as a subalgebra of $C([0, 1], M_{kn})$. Such algebras are stably projectionless and have trivial K -theory. Simple inductive limits of finite direct sums of Razak blocks are classified by tracial data (see [21] and also [22]), and as a consequence are UHF-stable, hence \mathcal{Z} -stable. (Alternatively, since such limits have nuclear dimension one, \mathcal{Z} -stability follows from [32].)

THEOREM 3.1. *Let A be a Razak block as in (3.1), and let $a, b \in A^+$. Then $d_U(a, b) = d_W(a, b)$.*

Proof. The proof is an adaptation of [8, Theorem 4.1.6] (see also [9]). By Remark 2.2, we may assume that, for some $\gamma > 0$, both a and b are constant on $[0, \gamma]$ and $[1 - \gamma, 1]$. Moreover, it suffices to show that, if $d_W(a, b) < r$ and $\varepsilon > 0$, then there is a unitary $w \in \mathcal{U}(\tilde{A})$ such that

$$\|w_t a_t w_t^* - b_t\| < r + \varepsilon \quad \text{for every } t \in [0, 1].$$

So let us take such r and ε . Let $\delta \in (0, \varepsilon/2)$. For every $t \in [0, 1]$, we have $d_W(a_t, b_t) \leq d_W(a, b) < r$, which means that the eigenvalues of a_t and b_t , listed in ascending order, are paired within distance r of each other. We will use this to construct w on $[0, \gamma]$, $[\gamma, 1 - \gamma]$ and $[1 - \gamma, 1]$.

On $[\gamma, 1 - \gamma]$: Let D_t^a (respectively D_t^b) be the diagonal matrix whose diagonal entries are the eigenvalues of a_t (respectively b_t) in ascending order. By [31, Lemma 1.1] and [31, Corollary 1.3], D^a and D^b are continuous, and there exist unitaries $u, v \in C([\gamma, 1 - \gamma], M_{kn})$ such that for every $t \in [\gamma, 1 - \gamma]$,

$$\|u_t a_t u_t^* - D_t^a\| < \delta \quad \text{and} \quad \|v_t b_t v_t^* - D_t^b\| < \delta. \quad (3.2)$$

Define

$$w(t) = v_t^* u_t \quad \text{for } t \in [\gamma, 1 - \gamma]. \quad (3.3)$$

Then for such t we have

$$\|w_t a_t w_t^* - b_t\| \leq \|D_t^a - D_t^b\| + 2\delta < r + \varepsilon.$$

On $[0, \gamma]$: Choose unitaries $U, V \in M_k$ such that, with $u_0 = (U \otimes 1_{n-1}) \oplus (1_k \otimes e_{nn})$ and $v_0 = (V \otimes 1_{n-1}) \oplus (1_k \otimes e_{nn})$, each M_k block of $u_0 a_0 u_0^*$ and $v_0 b_0 v_0^*$ is diagonal with the eigenvalues in ascending order. In particular,

$$\|u_0 a_0 u_0^* - v_0 b_0 v_0^*\| < r. \quad (3.4)$$

Choose a permutation matrix x such that $x u_0 a_0 u_0^* x^*$ and $x v_0 b_0 v_0^* x^*$ are diagonal matrices whose entries appear in ascending order (the same x works for both matrices).

That is, since $a_0 = a_\gamma$,

$$(x u_0) a_0 (x u_0)^* = D_\gamma^a \quad \text{and} \quad (x v_0) b_0 (x v_0)^* = D_\gamma^b. \quad (3.5)$$

Then

$$\|[(x u_0)^* u_\gamma, a_0]\| = \|u_\gamma a_0 u_\gamma^* - (x u_0) a_0 (x u_0)^*\| = \|u_\gamma a_\gamma u_\gamma^* - D_\gamma^a\| < \delta, \quad (3.6)$$

and similarly $\|[(x v_0)^* v_\gamma, b_0]\| < \delta$. By [19, Lemma 2.6.11], which is an exercise in functional calculus, provided we chose δ sufficiently small to begin with, there are therefore paths $f, g \in C([0, \gamma], \mathcal{U}_{kn})$ from 1 to $(x u_0)^* u_\gamma$ and $(x v_0)^* v_\gamma$ respectively, such that

$$\|[f_t, a_0]\| < \varepsilon/2 \quad \text{and} \quad \|[g_t, b_0]\| < \varepsilon/2 \quad \text{for every } t \in [0, \gamma]. \quad (3.7)$$

Define

$$w_t = g_t^* (x v_0)^* (x u_0) f_t \quad \text{for } t \in [0, \gamma]. \quad (3.8)$$

Note that $g_\gamma^*(xv_0)^*(xu_0)f_\gamma = v_\gamma^*u_\gamma$, so now w is well defined and continuous on $[0, 1 - \gamma]$. Moreover,

$$w_0 = v_0^*u_0 = (V^*U \otimes 1_{n-1}) \oplus (1_k \otimes e_{nn}). \quad (3.9)$$

Finally, for $t \in [0, \gamma]$ we have

$$\begin{aligned} \|w_t a_t w_t^* - b_t\| &= \|(xu_0)(f_t a_t f_t^*)(xu_0)^* - (xv_0)(g_t b_t g_t^*)(xv_0)^*\| \\ &\leq \|(xu_0)a_t(xu_0)^* - (xv_0)b_t(xv_0)^*\| + \varepsilon \\ &= \|u_0 a_0 u_0^* - v_0 b_0 v_0^*\| + \varepsilon \\ &< r + \varepsilon. \end{aligned}$$

On $[1 - \gamma, 1]$: By exactly the same argument, we extend w continuously to $[1 - \gamma, 1]$ with $\|w_t a_t w_t^* - b_t\| < r + \varepsilon$ for every t and with

$$w_1 = V^*U \otimes 1_n. \quad (3.10)$$

(This is because $U \otimes 1_n$ and $V \otimes 1_n$ play the diagonalising roles at $t = 1$ that $u_0 = (U \otimes 1_{n-1}) \oplus (1_k \otimes e_{nn})$ and $v_0 = (V \otimes 1_{n-1}) \oplus (1_k \otimes e_{nn})$ did at $t = 0$. A different permutation matrix x may be needed but this does not matter.) Comparing (3.9) and (3.10) we see that the unitary w we have constructed is indeed in \tilde{A} (because $w - 1 \in A$), so we are done. \square

COROLLARY 3.2. *If A is a sequential inductive limit of finite direct sums of Razak blocks, then $d_U(a, b) = d_W(a, b)$ for every $a, b \in A^+$.*

REMARK 3.3. In the unital case Theorem 3.1 holds for more general type I C^* -algebras. In particular, Cheong has shown in [9] that it is true for any one-dimensional NCCW complex.

4. SIMPLE, \mathcal{Z} -STABLE C^* -ALGEBRAS

The proof of our main theorem, that d_U and d_W agree for full spectrum positive contractions in simple \mathcal{Z} -stable C^* -algebras, relies on the structure of the Cuntz semi-group of these algebras and on the classification established in [22]. A preliminary discussion seems warranted.

Building on [6] and [7], it is shown in [13] that if A is simple, stably finite, \mathcal{Z} -stable and exact, then

$$\text{Cu}(A) \cong V(A) \setminus \{0\} \sqcup \text{LAff}(T(A)), \quad (4.1)$$

where $V(A)$ is the Murray–von Neumann semigroup of projections over A , $T(A)$ is the cone of densely finite lower semicontinuous traces on A , and $\text{LAff}(T(A))$ denotes the union of the zero functional and those lower semicontinuous linear functionals that are suprema of increasing sequences of elements in

$$\text{Aff}(T(A)) = \{f : T(A) \rightarrow \mathbb{R}^+ \mid f \text{ linear, continuous, } f|_{T(A) \setminus \{0\}} > 0\}.$$

(The isomorphism is obtained by sending $[a]$ to $[p] \in V(A)$ if a is Cuntz equivalent to a nonzero projection p , and otherwise to the functional $\widehat{[a]}$ on $T(A)$ defined by $\widehat{[a]}(\tau) = d_\tau(a) = \lim_{n \rightarrow \infty} \tau(a^{1/n})$. Note that if $e \in A$ is strictly positive, then $\widehat{[e]}(\tau) = \|\tau\|$ for every $\tau \in T(A)$.)

The main theorem of [22] is that the finer invariant Cu^\sim classifies $*$ -homomorphisms from inductive limits of one-dimensional NCCW complexes with trivial K_1 (a class of C^* -algebras that includes in particular those considered in Corollary 3.2) to stable rank one C^* -algebras. We refer the reader to [22] for the definition but note in particular that (in the stable rank one case), $\text{Cu}^\sim(A)$ contains $\text{Cu}(A)$ as its positive cone. If moreover A is simple and \mathcal{Z} -stable then it follows from (4.1) that

$$\text{Cu}^\sim(A) \cong K_0(A) \setminus \{0\} \sqcup \text{LAff}^\sim(T(A)) \quad (4.2)$$

where

$$\text{LAff}^\sim(T(A)) = \text{LAff}(T(A)) - \text{Aff}(T(A)) \quad (4.3)$$

(see [22, Section 6] for details).

THEOREM 4.1. *Let A be simple, separable, \mathcal{Z} -stable, exact and of stable rank one. Then $d_W(a, b) = d_U(a, b)$ for every $a, b \in A^+$ with $\sigma(a) = \sigma(b) = [0, 1]$.*

Proof. The argument is essentially the same as is laid out in [30, Section 3.4] (except that we do not have to worry about K -theory). By (4.1) and (4.2), we have

$$\text{Cu}(A) \cong V(A) \setminus \{0\} \sqcup \text{LAff}(T(A)), \quad \text{Cu}^\sim(A) \cong K_0(A) \setminus \{0\} \sqcup \text{LAff}^\sim(T(A)). \quad (4.4)$$

Let $s_A \in A$ be strictly positive. By [17, Proposition 5.3], there exists a simple inductive limit B of finite direct sums of Razak blocks together with a strictly positive element $s_B \in B$ and an isomorphism $\gamma : T(A) \rightarrow T(B)$ under which $\widehat{[s_A]}$ corresponds to $\widehat{[s_B]}$. (Essentially, B is the tensor product of the C^* -algebra \mathcal{W} , a simple monotracial inductive limit of Razak blocks, with an appropriate AF-algebra.) Since B is \mathcal{Z} -stable and $K_0(B) = 0$ (see Section 3), we have

$$\text{Cu}(B) = \text{LAff}(T(B)), \quad \text{Cu}^\sim(B) = \text{LAff}^\sim(T(B)). \quad (4.5)$$

The obvious morphism

$$\theta : \text{Cu}^\sim(B) \rightarrow \text{LAff}^\sim(T(A)) \subset \text{Cu}^\sim(A), \quad (4.6)$$

that maps $\text{Cu}(B)$ onto $\text{LAff}(T(A))$ via

$$\theta([b])(\tau) = \widehat{[b]}(\gamma(\tau)) = d_{\gamma(\tau)}(b) \quad (\tau \in T(A)), \quad (4.7)$$

satisfies $\theta([s_B]) = \widehat{[s_A]} \leq [s_A]$ and therefore, by [22, Theorem 1.0.1], lifts to an embedding $\iota : B \rightarrow A$ with $\text{Cu}^\sim(\iota) = \theta$.

Next, since no nonzero element of $C^*(a)$ is Cuntz equivalent to a projection, the inclusion $C^*(a) \hookrightarrow A$ induces a morphism

$$\text{Cu}(C^*(a)) \rightarrow \text{LAff}(T(A)) \subset \text{Cu}(A),$$

hence a morphism

$$\varphi : \text{Cu}(C_0(0, 1]) \rightarrow \text{Cu}(B), \quad (4.8)$$

namely

$$\varphi([f])(\tau) = d_{\gamma^{-1}(\tau)}(f(a)) \quad (f \in C_0(0, 1]^+, \tau \in T(B)), \quad (4.9)$$

that moreover satisfies $\varphi([\text{id}]) \leq [s_B]$. By [11, Theorem 1] we can lift this to a *-homomorphism $C_0(0, 1] \rightarrow B$. In other words, we can find a positive contraction $a' \in B$ such that

$$d_{\gamma(\tau)}(f(a')) = d_\tau(f(a)) \quad (f \in C_0(0, 1]^+, \tau \in T(A)). \quad (4.10)$$

Then for every $f \in C_0(0, 1]^+$ we have (in $\text{Cu}(A)$)

$$[\iota(f(a'))] = \theta([f(a')]) = [f(a)] \quad (4.11)$$

by (4.7) and (4.10).

Therefore, regarding B as a subalgebra of A , we have $d_W(a, a') = 0$. Since A has stable rank one, we have $d_U(a, a') = 0$ as well. Similarly we find a positive contraction $b' \in B$ with $d_U(b, b') = 0$. Then, specifying the algebra in which the relevant distance should be measured,

$$\begin{aligned} d_W^A(a, b) &= d_W^A(a', b') \\ &= d_W^B(a', b') \quad (4.7) \\ &= d_U^B(a', b') \quad (\text{Corollary 3.2}) \\ &\geq d_U^A(a', b') \quad (\mathcal{U}(\tilde{B}) \subset \mathcal{U}(\tilde{A})) \\ &= d_U^A(a, b), \end{aligned}$$

hence (see Remark 2.2), $d_W(a, b) = d_U(a, b)$. □

REMARK 4.2. We note here that the hypotheses of Theorem 4.1 can be relaxed.

- (i) A simple, unital, \mathcal{Z} -stable C*-algebra either has stable rank one or is purely infinite ([14] and [26, Theorem 6.7]). If A is unital and purely infinite, then $d_U(a, b) = 0$ by [29, Lemma 2.11] (or indeed from the very general [12, Theorem 1.7]) and $d_W(a, b) = 0$ more or less by definition. So, at least in the unital case, the assumption of stable rank one can be dropped. (A suitable generalisation of [26, Theorem 6.7], which says that stable rank one for simple, unital, stably finite, \mathcal{Z} -stable algebras is automatic, is not known for stably projectionless algebras.)
- (ii) By considering 2-quasitraces rather than traces, Theorem 4.1 is probably true without the assumption of exactness.
- (iii) Recall that a simple, separable C*-algebra is *pure* if its Cuntz semigroup has strict comparison and almost divisibility (see [35, Definition 3.6]). In particular, simple, separable, \mathcal{Z} -stable algebras are pure (see [26] and also [35, Proposition 3.7], which is stated for unital algebras but does not use this assumption). The computations (4.1) and (4.2) hold not just for the class of (simple, separable) \mathcal{Z} -stable algebras, but for the strictly larger class of pure C*-algebras that have stable rank one (see [20, Section 4] for details, and the references therein). So Theorem 4.1 holds for pure, stable rank one C*-algebras as well.

5. THE LÉVY–PROKHOROV DISTANCE

In this section we provide the C^* -analogue of [2, Theorem 1.3], namely, we compute the distance between the unitary orbits of self-adjoint elements in terms of spectral data. We first need a C^* -version of (1.2).

Let A be a unital, stably finite, exact C^* -algebra. For positive contractions $a, b \in A$, define $d_P(a, b)$ to be the infimum over $r > 0$ such that

$$\mu_{\tau, a}(U_r) \geq \mu_{\tau, b}(U) \quad \text{and} \quad \mu_{\tau, b}(U_r) \geq \mu_{\tau, a}(U) \quad (5.1)$$

for every open subset $U \subset (0, 1]$ and every trace $\tau \in T(A)$ (where $\mu_{\tau, a}$ and $\mu_{\tau, b}$ denote the Borel measures induced by τ on $C^*(a)$ and $C^*(b)$ respectively, and $U_r = \{t \mid \text{dist}(t, U) < r\}$).

Note that if $U \subset (0, 1]$ is open and the support of $f \in C_0(0, 1]^+$ is U , then $\mu_{\tau, a}(U) = d_\tau(f(a))$. In particular, $d_P(a, b) = 0$ if and only if $d_\tau(f(a)) = d_\tau(f(b))$ for every $\tau \in T(A)$ and $f \in C_0(0, 1]^+$.

The following is contained in Lemmas 1 and 2 of [9].

LEMMA 5.1. *Let A be unital, simple, exact and stably finite, and suppose moreover that A has strict comparison (for example, A might be \mathcal{Z} -stable). Then for every $a, b \in A^+$ with $\sigma(a) = \sigma(b) = [0, 1]$, we have*

$$d_W(a, b) \leq d_P(a, b) \leq d_U(a, b). \quad (5.2)$$

Proof. The inequality $d_P \leq d_U$ is proved exactly as in [24, Lemma 1], and no assumption on spectra is needed. The condition $\sigma(a) = [0, 1]$ ensures that no nonzero element of $C^*(a)$ is Cuntz equivalent to a projection (and similarly for b), and then the inequality $d_W \leq d_P$ is readily obtained by appealing to a version of strict comparison not available to projections, namely: $x \precsim y$ if and only if $d_\tau(x) \leq d_\tau(y)$ for every τ (see [1, Proposition 5.9]). \square

Combining Theorem 4.1 and Lemma 5.1, we have the following.

THEOREM 5.2. *Let A be unital, simple, separable, stably finite, \mathcal{Z} -stable and exact, and let $a, b \in A^+$ with $\sigma(a) = \sigma(b) = [0, 1]$. Then*

$$d_U(a, b) = d_P(a, b) = d_W(a, b).$$

In particular, a and b are approximately unitarily equivalent if and only if $d_\tau(f(a)) = d_\tau(f(b))$ for every $\tau \in T(A)$ and $f \in C_0(0, 1]^+$. This has been obtained without the assumption of simplicity in [23, Theorem 1.3].

6. OUTLOOK

Here are two avenues open for continued study.

- (i) What can be said of *normal* operators in a simple, \mathcal{Z} -stable C^* -algebra of stable rank one? Or in other words, full spectrum positive contractions represent injective $*$ -homomorphisms from $C_0(0, 1]$. How far can one extend our results from $[0, 1]$ to compact metric spaces X (compared for example to [12] and [29])? In light of what happens in the world of semifinite factors, one would expect, at best, inequalities.

- (ii) Extending another way, a positive contraction in A represents an ‘order zero’ map from \mathbb{C} to A . What can be said about order zero maps from, for example, finite dimensional algebras? This question will be addressed in subsequent work.

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